Tentamen Metrische Ruimten, 03/07/03

1. Let $\mathfrak{T}$ consist of all subsets U of the set of real numbers $\mathbb{R}$ such that $\mathbb{R} \setminus U$ is finite, together with the empty set $\emptyset$.

   (a) Show that $(\mathbb{R}, \mathfrak{T})$ is a topological space.

   (b) Is $(\mathbb{R}, \mathfrak{T})$ compact?

   (c) Is $(\mathbb{R}, \mathfrak{T})$ Hausdorff?

   Justify the answers!

2. Let S_1, S_2, T_1, T_2 be topological spaces, and let $f_i : S_i \to T_i$, $i = 1, 2$ be two functions. Define

$$f_1 \times f_2 : S_1 \times S_2 \to T_1 \times T_2$$

   by $(f_1 \times f_2)(x, y) := (f_1(x), f_2(y))$.

   (a) Prove that $f_1 \times f_2$ is continuous if and only if f_1 and f_2 are continuous.

   (b) Prove that if f_1 and f_2 are homeomorphisms then also $f_1 \times f_2$ is a homeomorphism.

3. Let the topological space T be Hausdorff. Show that for each point $x \in T$ the set $\{x\}$ is closed.

4. Determine the closure and the boundary of each of the following subsets of $\mathbb{R}$ with the usual Euklidian metric. Which of these sets are dense or nowhere dense in $\mathbb{R}$? ($\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ the set of integers.)

   (a) $\mathbb{R}$;

   (b) $\mathbb{Q} \cap [-1, 2)$;

   (c) $\{\frac{1}{n} : n \in \mathbb{N}\}$;

   (d) $\mathbb{R} \setminus \mathbb{Q}$;

   (e) $\mathbb{R} \setminus \mathbb{Z}$;

   (f) $\mathbb{Z}$.

5. If T is a connected topological space containing more than one point, and if $\{x\}$ is closed for every point $x \in T$, show that the number of points in T is infinite.